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One-Sided Approximation with Side Conditions

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A special case of the restricted range approximation scheme is the one-sided approximation scheme, introduced by Kammerer [5], which may be defined as

One-sided approximation scheme. For an $f \in C[a, b]$, approximate f by polynomials $p \in \Pi$ which always lie above f ($p(x) \geqslant f(x)$ for all $x \in [a, b]$).

The one-sided approximation scheme can be viewed as the imposition of a relatively simple type of nonlinear side condition upon the usual Chebyshev approximation process. In this paper we consider imposing an additional finite number of linear side conditions on the above approximation process and term the resulting scheme the one-sided approximation with side conditions (OSAS) scheme:

OSAS scheme. Suppose $x_1^*,..., x_n^*$ are n bounded linear functionals on $C[a, b]$. For an $f \in C[a, b]$, approximate f by polynomials $p \in \Pi$ which always lie above f and also interpolate f at the x_i^* (i.e., $x_i^*p = x_i^*f$, $i = 1,..., n$).

As usual, we shall say that we have a Weierstrass theorem holding for the OSAS scheme in case given $\epsilon > 0$ and $f \in C[a, b]$ arbitrary, it is possible to find a $p \in \Pi$ that not only lies above f and interpolates f at the $x_1^*,..., x_n^*$ but also is within ϵ of f (in the Chebyshev norm). Similarly, by a Jackson-type theorem, we mean a theorem relating the deviation from f of the best polynomial p of degree k that lies above f and interpolates f at the $x_1^*,..., x_n^*$ to the deviation from f of the best Chebyshev polynomial approximation q of degree k to f.

As is well known, a Weierstrass theorem holds for the usual unconstrained Chebyshev approximation process. Consequently, a Weierstrass theorem also holds for the classical one-sided approximation scheme itself. (Given $\epsilon > 0$ arbitrary, let $p \in \Pi$ be such that $\|f - p\| < \epsilon/2$. Then $q = p + \epsilon/2 \in \Pi$,

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lies above f, and $\|f - q\| < \epsilon$.) We likewise have a Jackson-type theorem for the one-sided approximation scheme, the deviation of the best one-sided polynomial approximation p to f of degree k being at most twice the deviation of the best Chebyshev polynomial approximation q to f of degree k.

As is fairly well known, if we impose a finite number of linear side conditions on the usual Chebyshev approximation process, we still have a Weierstrass theorem (the result being Yamabe's theorem [6]) and also a Jackson-type theorem (the bounded linear functional (BLFT) theorem [3]). On the other hand, if one imposes both linear and nonlinear side conditions on the usual Chebyshev approximation process, we need not always have even a Weierstrass theorem holding. This is perhaps best illustrated by the SAIN approximation scheme introduced by Deutsch and Morris [1], which may be defined as

SAIN approximation scheme. Suppose $x_1^*,..., x_n^*$ are n bounded linear functionals on $C[a, b]$. For an $f \in C[a, b]$, approximate f by polynomials $p \in \Pi$ whose norm are the same as that of f ($\|p\| = \|f\|$) and which also interpolate f at the x_i^*.

Notice that although the OSAS and SAIN approximation schemes are very similar to each other in their statements, neither is a special case of the other (the analogy can be extended even farther if it is recalled that for the related approximation scheme of approximating an $f \in C[a, b]$ by polynomials p whose norm is that of f, one also has both a Weierstrass and a Jackson-type theorem holding (e.g., [3]). Thus, SAIN without the linear side conditions exhibits similar behavior to the one-sided approximation scheme for Weierstrass and Jackson-type theorems). One the other hand, for the SAIN approximation scheme, it is well known that whether a Weierstrass theorem holds depends heavily on the particular linear functionals $x_1^*,..., x_n^*$ involved. Thus, one might suspect a similar behavior for the OSAS approximation scheme. That such is indeed the case is pointed out forcibly by the theorem of [4], where necessary and sufficient conditions on an $f \in C[a, b]$ are given in order that one even have a polynomial $p \in \Pi$ existing that lies above f and interpolates f at a given number of points; in particular, even for the $x_1^*,..., x_n^*$ all point evaluations on $C[a, b]$ one does not have a Weierstrass theorem holding for the OSAS approximation scheme (this should be constrasted with the SAIN approximation scheme, where the $x_1^*,..., x_n^*$ point evaluations do suffice for a Weierstrass theorem to hold).

It is well known that a complete determination of what linear functionals $x_1^*,..., x_n^*$ are necessary and sufficient for a Weierstrass theorem to hold has not as yet been given, partially because the problem seems inherently difficult. It was this fact (and the fact that the author was looking at the more general problem of obtaining a Jackson-type theorem for the more general restricted range approximation scheme, together with the fact that the author has previously obtained a Jackson-type theorem for the SAIN approximation scheme when the x_i^* were all point evaluations) that led the author to consider obtaining necessary and sufficient conditions on the $x_1^*,..., x_n^*$ in order that a Weierstrass theorem hold for the OSAS approximation scheme. While a priori it was not at all clear that such could be done, it turns out that the OSAS scheme is so simple that we can even give a Jackson-type theorem as a corollary.

We will require the following known result (e.g., [2, pp. 86–87]):

PROPOSITION A. *Let X be a normed linear space, $\{c_1,..., c_n\}$ arbitrary scalars, $\{x_1^*,..., x_n^*\}$ a finite set in X^*, and let $M > 0$. Then, for any $\epsilon > 0$ there exists an $x \in X$ such that $x_i^*(x) = c_i$, $i = 1,..., n$, and $\| x \| < M + \epsilon$ if and only if*

$$| \Sigma \alpha_i c_i | \leqslant M \| \Sigma \alpha_i x_i^* \|$$

for every finite collection of scalars $\{\alpha_i\}$.

DEFINITION 1. We say that a finite set of bounded linear functionals $x_1^*,..., x_n^*$ are *span indefinite* in case no nontrivial linear combination of the $x_1^*,..., x_n^*$ is a positive linear functional.

Equivalently, if $\mathscr{P}$ denotes the cone of positive linear functionals and $\langle x_1^*,..., x_n^* \rangle$ the subspace of the dual spanned by $x_1^*,..., x_n^*$, then the $x_1^*,..., x_n^*$ are span indefinite if and only if

$$\langle x_1^*,..., x_n^* \rangle \cap \mathscr{P} = \{0\},$$

if and only if

$$\langle x_1^*,..., x_n^* \rangle \cap (\mathscr{P} \cup -\mathscr{P}) = \{0\},$$

$-\mathscr{P}$ being $\{-p; p \in \mathscr{P}\}$. Notice that $x_1^*,..., x_n^*$ being span indefinite implies, in particular, that they are linearly independent. On the other hand, if the linear side conditions $x_1^*,..., x_n^*$ imposed in the OSAS scheme are not linearly independent, they can be replaced by a subset that is linearly independent, so without loss of generality, suppose below that the $x_1^*,..., x_n^*$ are linearly independent.

LEMMA 1. *Suppose that*

(i) *$x_1^*,..., x_n^*$ are span indefinite on a function space X,*

(ii) *$Y = \{x \in X; x_2^* x = \cdots = x_n^* x = 0\}$, and*

(iii) *$\mathbf{1} \in Y$.*

Then $\pm x_1^* |_Y$ is not a positive linear functional on Y.

Proof. Suppose not. By the Hahn–Banach theorem, let $u^* \in X^*$ be an extension of $x_1^* |_Y$ to X such that $\| u^* \| = \| x_1^* |_Y \|$. Let $w^* \in X^*$ be such that $x_1^* = u^* + w^*$. For $y \in Y$, $x_1^* y = u^* y$, so $w^* y = 0$ ($y \in Y$), whence, $w^* \in Y^\perp = \langle x_2^*,..., x_n^* \rangle$. On the other hand, $| u^* \mathbf{1} | = | x_1^* |_Y (\mathbf{1})| = \| x_1^* |_Y \| = \| u^* \|$, so u^* is ($\pm$) a positive linear function. But $u^* = x_1^* - w^* \in \langle x_1^*,..., x_n^* \rangle$, whence, by the span indefiniteness of $x_1^*,..., x_n^*$, u^* cannot be ($\pm$) a positive linear functional. ∎

LEMMA 2. *Suppose that M is a dense subspace of $C[a, b]$ that contains the constants. If $x_1^*,..., x_n^*$ are span indefinite on $C[a, b]$, then there exists an $m \in M$ such that*

(i) $m(x) \geqslant 1$, $(x \in [a, b])$, *and*

(ii) $x_i^* m = 0$, $(i = 1,..., n)$.

Proof. Clearly, it suffices to find an $m \in M$ that satisfies condition (ii) and is strictly positive on $[a, b]$. By Yamabe's theorem [2, p. 87; 6] it suffices to find an $x \in X = C[a, b]$ that is zero at the x_i^* ($x_i^* x = 0$, $i = 1,..., n$) and strictly positive on $[a, b]$.

If $x_i^* \mathbf{1} = 0$ ($i = 1,..., n$), done, so without loss of generality suppose that $x_1^* \mathbf{1} \neq 0$. By replacing x_1^* by $y_1^* = (\operatorname{sgn} x_1^* \mathbf{1})\, x_1^* / \| x_1^* \|$ and x_j^* by $y_j^* = x_j^* - (x_j^* \mathbf{1} / x_1^* \mathbf{1})\, x_1^*$, without loss of generality we may suppose that

$$x_1^* \mathbf{1} > 0, \qquad \| x_1^* \| = 1,$$

while

$$x_2^* \mathbf{1} = \cdots = x_n^* \mathbf{1} = 0.$$

By Lemma 1, $| x_1^* \mathbf{1} | < 1$, whence, by Proposition A, (applied to the subspace Y of Lemma 1 and $x_1^* |_Y$) there is a $g \in X$ such that $x_1^* g = -x_1^* \mathbf{1}$, $x_2^* g = \cdots = x_n^* g = 0$, and $\| g \| < 1$. But then $x = \mathbf{1} + g > 0$ on $[a, b]$ and is zero at each of the x_i^*, $i = 1,..., n$. ∎

THEOREM 1. *Suppose $x_1^*,..., x_n^*$ are linearly independent bounded linear functionals on $C[a, b]$. Then, for any $f \in C[a, b]$ and $\epsilon > 0$, there is a $p \in \Pi$ for which*

(i) $p(x) \geqslant f(x)$, $x \in [a, b]$),

(ii) $x_i^* p = x_i^* f$, $(i = 1,..., n)$, *and*

(iii) $\| f - p \| < \epsilon$,

if and only if the $x_1^*,..., x_n^*$ are span indefinite on $C[a, b]$.

Proof. Suppose first that the $x_1^*,..., x_n^*$ are span indefinite on $C[a, b]$. By Lemma 2 let $m \in \Pi$ be such that $x_i^*(m) = 0$ ($i = 1,..., n$), while $m(x) \geqslant 1$

on $[a, b]$. Let $\mu = \| m \|$. Given $f \in C[a, b]$ and $\epsilon > 0$ arbitrary, let $q \in \Pi$ be such that $x_i^*q = x_i^*f$ $(i = 1,..., n)$ and $\|f - q\| < \epsilon/(1 + \mu)$. Set

$$p = q + \epsilon m/(1 + \mu).$$

Then $p \in \Pi$, $x_i^*p = x_i^*f$ $(i = 1,..., n)$, and $\|f - p\| \leqslant \|f - q\| + \epsilon\mu/(1 + \mu) < \epsilon$.

Conversely, suppose that $x_1^*,..., x_n^*$ are not span indefinite on $C[a, b]$. In particular, then, there exists constants $\xi_1,..., \xi_n$ such that $x^* = \xi_1 x_1^* + \cdots + \xi_n x_n^*$ is a positive linear functional of norm one on $X = C[a, b]$. Using the Riesz representation theorem, let

$$x^*(\cdot) = \int_a^b \cdot \, d\mu,$$

μ being a finite nonnegative Baire measure on $[a, b]$. Let $\mu = \mu_1 + \mu_2$, where μ_1 is purely atomic and μ_2 has no atoms. If $y_i^* = \int d\mu_i$, then in order for a $p \in \Pi$ to be such that $x^*p = x^*f$, we must also have $y_i^*p = y_i^*f$ $(i = 1, 2)$. If $\mu_1 \neq 0$, it consists of at most a countable number of point evaluations, say at $\{t_i\}_{i=1}^{\infty}$. But a $p \in \Pi$ will interpolate f at y_1^* only if $p(t_i) = f(t_i)$ for every t_i. Hence, we need merely take an $f \in C[a, b]$ that fails the necessary condition at t_1 in the theorem of [4] mentioned above to get a f that cannot be approximated arbitrarily closely by polynomials in the OSAS scheme. If $\mu_1 = 0$ but $\mu_2 \neq 0$, then the support of μ_2 has positive Lebesque measure. Pick a point t_0 in $[a, b]$ and consider $\nu = \chi_{[a,b]\setminus\{t_0\}} \circ \mu_2$. Let A be a closed subset of the support of μ_2 disjoint from $\{t_0\}$ that has positive Lebesque measure. Since $\| \nu \| = \| \mu_2 \|$ it is possible to do so such that $\mu = \chi_A \circ \mu_2 = \chi_A \circ \nu$ is nonzero on $C[a, b]$. Define a continuous function f, $0 \leqslant f(t) \leqslant 1$, by Urysohn's lemma so that $f(x) = 1$, $x \in A$, but $f(t_0) = 0$. Then for any polynomial p to satisfy the OSAS scheme (and in particular that $y_2^*p = y_2^*f$) it is necessary that $p(x) = 1$ for all $x \in A$. But A has positive measure, hence, necessarily $p(x) \equiv 1$. Thus, $\|f - p\| \geqslant |f(t_0) - p(t_0)| = 1$, and again, we fail to have a Weierstrass theorem holding. ∎

From the proof, we observe that we have also established the following:

COROLLARY 1. *Suppose that $x_1^*,..., x_n^*$ are span indefinite on $X = C[a, b]$. Let M be a dense subspace of X that contains the constants. Then, given $x \in X$ and $\epsilon > 0$ arbitrary, there exist $m \in M$ such that*

(i) $m \geqslant x$,

(ii) $x_i^*m = x_i^*x$,

(iii) $\| x - m \| < \epsilon$.

COROLLARY 2. *If $x_1^*,\dots, x_n^*$ are span indefinite on $X = C[a, b]$, and if $\delta_k(f)$ denotes the deviation of the best Chebyshev polynomial of degree k to f, then there exists a constant C depending only on the $x_1^*,\dots, x_n^*$ such that, for any $f \in C[a, b]$, there is a polynomial $p_k \in \Pi_k$ of degree at most k for which*

(i) $p_k(x) \geqslant f(x)$, $(x \in [a, b])$

(ii) $x_i^* p_k = x_i^* f$, $(i = 1,\dots, n)$, *and*

(iii) $\| f - p_k \| \leqslant C\delta_k(f)$.

Also, a similar variant to Corollary 1 clearly can be given. However, the converse of Corollary 1 is not valid for arbitrary dense subspaces of $X = C[a, b]$ that contain the constants. As an example, consider the subspace

$$M = \{m \in C[0, 2];\ m \text{ agrees with a polynomial on } [0, 1]\}. \tag{1}$$

Since Π is dense in $C[0, 2]$ and is a subspace of M, M is a dense subspace of $C[0, 2]$ that contains the constants. Furthermore, the linear functional

$$x^* = e_{3/2}, \tag{2}$$

being a point evaluation at a point of $[0, 2]$ near which M is locally all continuous functions, will be such that a Weierstrass theorem will hold for the OSAS scheme.

On the other hand, for the converse to Corollary 1 to fail, it is clear that the type of behavior illustrated by (1) and (2) above must be occurring. In particular, for M any dense subspace of Π that contains the constant functions, the converse to Corollary 1 is valid, and we immediately can write down a Muntz-type theorem analogous to Theorem 1 if we wished to do so.

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